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BOUNDEDNESS IN FUNCTIONAL PERTURBED DIFFERENTIAL SYSTEMS

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Abstract. This paper shows that the solutions to the perturbed differential system

$$y' = f(t,y) + \int_{t_0}^{t} g(s,y(s))ds + h(t,y(t),Ty(t))$$

have bounded property. To show this property, we impose conditions on the perturbed part $\int_{t_0}^{t} g(s,y(s))ds, h(t,y(t),Ty(t))$, and on the fundamental matrix of the unperturbed system $y' = f(t,y)$.

1. Introduction and preliminaries

We consider the nonlinear differential system

$$x'(t) = f(t,x(t)), \quad x(t_0) = x_0, \tag{1.1}$$

where $f \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $\mathbb{R}^+ = [0,\infty)$ and $\mathbb{R}^n$ is the Euclidean n-space. We assume that the Jacobian matrix $f_x = \partial f/\partial x$ exists and is continuous on $\mathbb{R}^+ \times \mathbb{R}^n$ and $f(t,0) = 0$. Also, we consider functional perturbed differential system of (1.1)

$$y' = f(t,y) + \int_{t_0}^{t} g(s,y(s))ds + h(t,y(t),Ty(t)),\ y(t_0) = y_0, \tag{1.2}$$

where $g \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $h \in C(\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$, $g(t,0) = 0$, $h(t,0,0) = 0$, and $T : C(\mathbb{R}^+, \mathbb{R}^n) \to C(\mathbb{R}^+, \mathbb{R}^n)$ is a continuous operator.

For $x \in \mathbb{R}^n$, let $|x| = (\sum_{j=1}^{n} x_j^2)^{1/2}$. For an $n \times n$ matrix A, define the norm $|A|$ of A by $|A| = \sup_{|x|\le 1} |Ax|$.

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Let $x(t, t_0, x_0)$ denote the unique solution of (1.1) with $x(t_0, t_0, x_0) = x_0$, existing on $[t_0, \infty)$. Then we can consider the associated variational systems around the zero solution of (1.1) and around $x(t)$, respectively,

$$v'(t) = f_x(t, 0)v(t), \; v(t_0) = v_0 \tag{1.3}$$

and

$$z'(t) = f_x(t, x(t, t_0, x_0))z(t), \; z(t_0) = z_0. \tag{1.4}$$

The fundamental matrix $\Phi(t, t_0, x_0)$ of (1.4) is given by

$$\Phi(t, t_0, x_0) = \frac{\partial}{\partial x_0} x(t, t_0, x_0),$$

and $\Phi(t, t_0, 0)$ is the fundamental matrix of (1.3).

We recall some notions of h-stability [15].

Definition 1.1. The system (1.1) (the zero solution $x = 0$ of (1.1)) is called an *h-system* if there exist a constant $c \geq 1$, and a positive continuous function h on $\mathbb{R}^+$ such that

$$|x(t)| \leq c\,|x_0|\,h(t)\,h(t_0)^{-1}$$

for $t \geq t_0 \geq 0$ and $|x_0|$ small enough (here $h(t)^{-1} = \frac{1}{h(t)}$).

Definition 1.2. The system (1.1) (the zero solution $x = 0$ of (1.1)) is called

(hS)*h-stable* if there exists $\delta > 0$ such that (1.1) is an h-system for $|x_0| \leq \delta$ and h is bounded.

Pinto[14,15] introduced the notion of h-stability (hS) which is the intention of obtaining results about stability for a weakly stable system (at least, weaker than those given exponential asymptotic stability) under some perturbations. That is, Pinto extended the study of exponential asymptotic stability to a variety of reasonable systems called h-systems. Choi, Ryu [5] and Choi, Koo, and Ryu [6] investigated bounds of solutions for nonlinear perturbed systems. Also, Goo [8,9,10] and Goo et al. [3] investigated boundedness of solutions for nonlinear perturbed systems.

Let $\mathcal{M}$ denote the set of all $n \times n$ continuous matrices $A(t)$ defined on $\mathbb{R}^+$ and $\mathcal{N}$ be the subset of $\mathcal{M}$ consisting of those nonsingular matrices $S(t)$ that are of class C^1 with the property that $S(t)$ and $S^{-1}(t)$ are bounded. The notion of t_∞-similarity in $\mathcal{M}$ was introduced by Conti [7].

Definition 1.3. A matrix $A(t) \in \mathcal{M}$ is t_∞*-similar* to a matrix $B(t) \in \mathcal{M}$ if there exists an $n \times n$ matrix $F(t)$ absolutely integrable over $\mathbb{R}^+$, i.e.,

$$\int_0^\infty |F(t)|dt < \infty$$

such that

$$\dot{S}(t) + S(t)B(t) - A(t)S(t) = F(t) \tag{1.5}$$

for some $S(t) \in \mathcal{N}$.

The notion of t_∞-similarity is an equivalence relation in the set of all $n \times n$ continuous matrices on $\mathbb{R}^+$, and it preserves some stability concepts [7, 12].

The aim of this paper is to obtain some results on boundedness of the nonlinear functional differential systems under suitable conditions on perturbed term using the notion of t_∞-similarity.

We give some related properties that we need in the sequal.

Lemma 1.4. [15] *The linear system*

$$x' = A(t)x, \;\; x(t_0) = x_0, \tag{1.6}$$

where $A(t)$ is an $n \times n$ continuous matrix, is an h-system (respectively h-stable) if and only if there exist $c \geq 1$ and a positive and continuous (respectively bounded) function h defined on $\mathbb{R}^+$ such that

$$|\phi(t, t_0)| \leq c\, h(t)\, h(t_0)^{-1} \tag{1.7}$$

for $t \geq t_0 \geq 0$, where $\phi(t, t_0)$ is a fundamental matrix of (1.6).

We need Alekseev formula to compare between the solutions of (1.1) and the solutions of perturbed nonlinear system

$$y' = f(t, y) + g(t, y), \;\; y(t_0) = y_0, \tag{1.8}$$

where $g \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$ and $g(t, 0) = 0$. Let $y(t) = y(t, t_0, y_0)$ denote the solution of (1.8) passing through the point (t_0, y_0) in $\mathbb{R}^+ \times \mathbb{R}^n$.

The following is a generalization to nonlinear system of the variation of constants formula due to Alekseev [1].

Lemma 1.5. [2] *Let x and y be a solution of (1.1) and (1.8), respectively. If $y_0 \in \mathbb{R}^n$, then for all $t \geq t_0$ such that $x(t, t_0, y_0) \in \mathbb{R}^n$, $y(t, t_0, y_0) \in \mathbb{R}^n$,*

$$y(t, t_0, y_0) = x(t, t_0, y_0) + \int_{t_0}^{t} \Phi(t, s, y(s))\, g(s, y(s))\, ds.$$

THEOREM 1.6. [5] *If the zero solution of (1.1) is hS, then the zero solution of (1.3) is hS.*

THEOREM 1.7. [6] *Suppose that* $f_x(t,0)$ *is* t_∞*-similar to* $f_x(t,x(t,t_0,x_0))$ *for* $t \geq t_0 \geq 0$ *and* $|x_0| \leq \delta$ *for some constant* $\delta > 0$. *If the solution* $v = 0$ *of (1.3) is hS, then the solution* $z = 0$ *of (1.4) is hS.*

LEMMA 1.8. (*Bihari − type inequality*) *Let* $u, \lambda \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$ *and* $w(u)$ *be nondecreasing in* u. *Suppose that, for some* $c > 0$,

$$u(t) \leq c + \int_{t_0}^{t} \lambda(s) w(u(s)) ds, \ t \geq t_0 \geq 0.$$

Then

$$u(t) \leq W^{-1}\Big[W(c) + \int_{t_0}^{t} \lambda(s) ds\Big],$$

where $t_0 \leq t < b_1$, $W(u) = \int_{u_0}^{u} \frac{ds}{w(s)}$, $W^{-1}(u)$ *is the inverse of* $W(u)$, *and*

$$b_1 = \sup\Big\{t \geq t_0 : W(c) + \int_{t_0}^{t} \lambda(s) ds \in \mathrm{domW}^{-1}\Big\}.$$

LEMMA 1.9. [3] *Let* $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$ *and* $w(u)$ *be nondecreasing in* u, $u \leq w(u)$. *Suppose that for some* $c > 0$ *and* $0 \leq t_0 \leq t$

$$\begin{aligned} u(t) \leq c &+ \int_{t_0}^{t} \lambda_1(s) u(s) ds + \int_{t_0}^{t} \lambda_2(s) w(u(s)) ds \\ &+ \int_{t_0}^{t} \lambda_3(s) \int_{t_0}^{s} \lambda_4(\tau) u(\tau) d\tau ds + \int_{t_0}^{t} \lambda_5(s) \int_{t_0}^{s} \lambda_6(\tau) w(u(\tau)) d\tau ds. \end{aligned}$$

Then

$$\begin{aligned} u(t) \leq W^{-1}\Big[W(c) &+ \int_{t_0}^{t} (\lambda_1(s) + \lambda_2(s) + \lambda_3(s) \int_{t_0}^{s} \lambda_4(\tau) d\tau \\ &+ \lambda_5(s) \int_{t_0}^{s} \lambda_6(\tau) d\tau) ds\Big], \end{aligned}$$

where $t_0 \leq t < b_1$, W, W^{-1} *are the same functions as in Lemma 1.8, and*

$$\begin{aligned} b_1 = \sup\Big\{t \geq t_0 : W(c) &+ \int_{t_0}^{t} (\lambda_1(s) + \lambda_2(s) + \lambda_3(s) \int_{t_0}^{s} \lambda_4(\tau) d\tau \\ &+ \lambda_5(s) \int_{t_0}^{s} \lambda_6(\tau) d\tau) ds \in \mathrm{domW}^{-1}\Big\}. \end{aligned}$$

For the proof we need the following corollary from Lemma 1.9.

Corollary 1.10. *Let* $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4 \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$ *and* $w(u)$ *be nondecreasing in* u, $u \le w(u)$. *Suppose that for some* $c > 0$ *and* $0 \le t_0 \le t$,

$$\begin{aligned} u(t) \le c + \int_{t_0}^{t} \lambda_1(s)u(s)ds + \int_{t_0}^{t} \lambda_2(s)w(u(s))ds \\ + \int_{t_0}^{t} \lambda_3(s) \int_{t_0}^{s} \lambda_4(\tau)u(\tau)d\tau ds. \end{aligned}$$

Then

$$u(t) \le W^{-1}\Big[W(c) + \int_{t_0}^{t} (\lambda_1(s) + \lambda_2(s) + \lambda_3(s) \int_{t_0}^{s} \lambda_4(\tau)d\tau)ds\Big],$$

where $t_0 \le t < b_1$, W, W^{-1} *are the same functions as in Lemma 1.8, and*

$$\begin{aligned} b_1 = \sup\Big\{t \ge t_0 : W(c) + \int_{t_0}^{t} (\lambda_1(s) + \lambda_2(s) \\ + \lambda_3(s) \int_{t_0}^{s} \lambda_4(\tau)d\tau)ds \in \mathrm{domW}^{-1}\Big\}. \end{aligned}$$

Lemma 1.11. [4] *Let* $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7 \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$, *and* $w(u)$ *be nondecreasing in* u, $u \le w(u)$. *Suppose that for some* $c > 0$ *and* $0 \le t_0 \le t$,

$$\begin{aligned} u(t) \le c + \int_{t_0}^{t} \lambda_1(s)u(s)ds + \int_{t_0}^{t} \lambda_2(s) \int_{t_0}^{s} (\lambda_3(\tau)w(u(\tau)) \\ + \lambda_4(\tau) \int_{t_0}^{\tau} \lambda_5(r)w(u(r))dr)d\tau ds + \int_{t_0}^{t} \lambda_6(s) \int_{t_0}^{s} \lambda_7(\tau)u(\tau)d\tau ds. \end{aligned}$$

Then

$$\begin{aligned} u(t) \quad &\le W^{-1}\Big[W(c) + \textstyle\int_{t_0}^{t} (\lambda_1(s) + \lambda_2(s) \int_{t_0}^{s} (\lambda_3(\tau) + \lambda_4(\tau) \int_{t_0}^{\tau} \lambda_5(r)dr)d\tau \\ &\qquad + \lambda_6(s) \textstyle\int_{t_0}^{s} \lambda_7(\tau)d\tau)ds\Big], \end{aligned}$$

where $t_0 \le t < b_1$, W, W^{-1} *are the same functions as in Lemma 1.8, and*

$$\begin{aligned} b_1 = \quad &\sup\Big\{t \ge t_0 : W(c) + \textstyle\int_{t_0}^{t} (\lambda_1(s) + \lambda_2(s) \int_{t_0}^{s} (\lambda_3(\tau) + \lambda_4(\tau) \int_{t_0}^{\tau} \lambda_5(r)dr)d\tau \\ &\qquad + \lambda_6(s) \textstyle\int_{t_0}^{s} \lambda_7(\tau)d\tau)ds \in \mathrm{domW}^{-1}\Big\}. \end{aligned}$$

2. Main results

In this section, we investigate boundedness for solutions of the functional perturbed differential systems via t_∞-similarity.

We need the following lemma to prove Theorem 2.2.

Lemma 2.1. *Let* $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7 \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$ *and* $w(u)$ *be nondecreasing in* u, $u \le w(u)$. *Suppose that, for some* $c \ge 0$ *and* $t \ge t_0$, *we have*

$$\begin{aligned} u(t) \le c + \int_{t_0}^{t} \lambda_1(s)w(u(s))ds + \int_{t_0}^{t} \lambda_2(s)\Big(\int_{t_0}^{s} (\lambda_3(\tau)u(\tau) \\ + \lambda_4(\tau)\int_{t_0}^{\tau} \lambda_5(s)w(u(r))dr)d\tau + \lambda_6(s)\int_{t_0}^{s} \lambda_7(\tau)u(\tau)d\tau\Big)ds. \end{aligned} \tag{2.1}$$

Then

$$\begin{aligned} u(t) \le W^{-1}\Big[W(c) + \int_{t_0}^{t} [\lambda_1(s) + \lambda_2(s)\Big(\int_{t_0}^{s} (\lambda_3(\tau) + \lambda_4(\tau)\int_{t_0}^{\tau} \lambda_5(r)dr)d\tau \\ + \lambda_6(s)\int_{t_0}^{s} \lambda_7(\tau)d\tau\Big)]ds\Big],\ t \ge t_0. \end{aligned} \tag{2.2}$$

Proof. Define a function $v(t)$ by the right member of (2.1). Then, we have $v(t_0) = c$ and

$$\begin{aligned} v'(t) &= \lambda_1(t)w(u(t)) + \lambda_2(t)\Big(\int_{t_0}^{t} (\lambda_3(s)u(s) \\ &\quad + \lambda_4(s)\int_{t_0}^{s} \lambda_5(\tau)w(u(\tau))d\tau)ds + \quad \lambda_6(t)\int_{t_0}^{t} \lambda_7(s)u(s)ds\Big) \\ &\le \Big[\lambda_1(t) + \lambda_2(t)\Big(\int_{t_0}^{t} (\lambda_3(s) + \lambda_4(s)\int_{t_0}^{s} \lambda_5(\tau)d\tau)ds \\ &\quad + \lambda_6(t)\int_{t_0}^{t} \lambda_7(s)ds\Big)\Big]w(v(t)), \end{aligned}$$

$t \ge t_0$, since $v(t)$ is nondecreasing, $u \le w(u)$, and $u(t) \le v(t)$. Now, by integrating the above inequality on $[t_0, t]$ and $v(t_0) = c$, we have

$$\begin{aligned} v(t) \le c + \int_{t_0}^{t} \Big(\lambda_1(s) + \lambda_2(s)\int_{t_0}^{s} (\lambda_3(\tau) + \lambda_4(\tau)\int_{t_0}^{\tau} \lambda_5(r)dr)d\tau \\ + \lambda_6(s)\int_{t_0}^{s} \lambda_7(\tau)d\tau\Big)w(z(s))ds. \end{aligned} \tag{2.3}$$

Thus, (2.3) yields the estimate (2.2). □

To obtain the bounded result, the following assumptions are needed:

(H1) $f_x(t,0)$ is t_∞-similar to $f_x(t,x(t,t_0,x_0))$ for $t \ge t_0 \ge 0$ and $|x_0| \le \delta$ for some constant $\delta > 0$.

(H2) The solution $x = 0$ of (1.1) is hS with the increasing function h.

(H3) $w(u)$ be nondecreasing in u such that $u \le w(u)$ and $\frac{1}{v}w(u) \le w(\frac{u}{v})$ for some $v > 0$.

THEOREM 2.2. *Let $a, b, c, k, q, u, w \in C(\mathbb{R}^+)$. Suppose that (H1), (H2), (H3), and g in (1.2) satisfies*

$$|g(t,y(t))| \le a(t)|y(t)| + b(t)\int_{t_0}^t k(s)w(|y(s)|)ds \tag{2.4}$$

and

$$|h(t,y(t),Ty(t))| \le c(t)(w(|y(t)|) + |Ty(t)|), |Ty(t)| \le \int_{t_0}^t q(s)|y(s)|ds, \tag{2.5}$$

where $t \ge t_0 \ge 0$, $\int_{t_0}^\infty a(s)ds < \infty$, $\int_{t_0}^\infty b(s)ds < \infty$, $\int_{t_0}^\infty c(s)ds < \infty$, $\int_{t_0}^\infty k(s)ds < \infty$, and $\int_{t_0}^\infty q(s)ds < \infty$. Then, any solution $y(t) = y(t,t_0,y_0)$ of (1.2) is bounded on on $[t_0,\infty)$ and it satisfies

$$|y(t)| \le h(t)W^{-1}\Big[W(c) + c_2\int_{t_0}^t [c(s) + \int_{t_0}^s (a(\tau) + b(\tau)\int_{t_0}^\tau k(r)dr)d\tau + c(s)\int_{t_0}^s q(\tau)d\tau]ds\Big],$$

where W, W^{-1} are the same functions as in Lemma 1.8, and

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + c_2\int_{t_0}^t [c(s) + \int_{t_0}^s (a(\tau) + b(\tau)\int_{t_0}^\tau k(r)dr)d\tau + c(s)\int_{t_0}^s q(\tau)d\tau]ds \in \mathrm{domW^{-1}}\Big\}.$$

Proof. Let $x(t) = x(t,t_0,y_0)$ and $y(t) = y(t,t_0,y_0)$ be solutions of (1.1) and (1.2), respectively. By Theorem 1.6, since the solution $x = 0$ of (1.1) is hS, the solution $v = 0$ of (1.3) is hS. Therefore, from (H1), by Theorem 1.7, the solution $z = 0$ of (1.4) is hS. Applying the nonlinear variation of constants formula , the hS condition of $x = 0$ of (1.1), together with (2.4) and (2.5), we have

$$
\begin{aligned}
&|y(t)|\\
&\leq |x(t)| + \int_{t_0}^{t} |\Phi(t,s,y(s))|(\int_{t_0}^{s} |g(\tau,y(\tau))|d\tau + |h(s,y(s),Ty(s))|)ds\\
&\leq c_1|y_0|h(t)\,h(t_0)^{-1} + \int_{t_0}^{t} c_2h(t)h(s)^{-1}\Big(\int_{t_0}^{s}(a(\tau)|y(\tau)|\\
&\quad + b(\tau)\int_{t_0}^{\tau} k(r)w(|y(r)|)dr)d\tau + c(s)(w(|y(s)|) + \int_{t_0}^{s} q(\tau)|y(\tau)|d\tau)\Big)ds.
\end{aligned}
$$

By the assumptions (H2) and (H3), we obtain

$$
\begin{aligned}
|y(t)| \leq c_1|y_0|h(t)\,h(t_0)^{-1} &+ \int_{t_0}^{t} c_2h(t)\Big(c(s)w(\frac{|y(s)|}{h(s)})\\
&+ \int_{t_0}^{s}(a(\tau)\frac{|y(\tau)|}{h(\tau)} + b(\tau)\int_{t_0}^{\tau} k(r)w(\frac{|y(r)|}{h(r)})dr)d\tau\\
&+ c(s)\int_{t_0}^{s} q(\tau)\frac{|y(\tau)|}{h(\tau)}d\tau\Big)ds.
\end{aligned}
$$

Set $u(t) = |y(t)||h(t)|^{-1}$. Then, by Lemma 2.1, we have

$$
\begin{aligned}
|y(t)| \leq h(t)W^{-1}\Big[W(c) + c_2\int_{t_0}^{t}[c(s) + \int_{t_0}^{s}(a(\tau) + b(\tau)\int_{t_0}^{\tau} k(r)dr)d\tau\\
+ c(s)\int_{t_0}^{s} q(\tau)d\tau]ds\Big],
\end{aligned}
$$

where $c = c_1|y_0|\,h(t_0)^{-1}$. The above estimation yields the desired result since the function h is bounded, and so the proof is complete. □

Remark 2.3. Letting $c(t) = 0$ in Theorem 2.2, we obtain the similar result as that of Theorem 3.4 in [8].

Theorem 2.4. *Let $a, b, k, q, u, w \in C(\mathbb{R}^+)$. Suppose that (H1), (H2), (H3), and g in (1.2) satisfies*

$$
\int_{t_0}^{s} |g(\tau,y(\tau))|d\tau \leq a(s)|y(s)| + b(s)\int_{t_0}^{s} k(\tau)|y(\tau)|d\tau \tag{2.6}
$$

and

$$
|h(t,y(t),Ty(t))| \leq b(t)(w(|y(t)|) + |Ty(t)|), |Ty(t)| \leq \int_{t_0}^{t} q(s)|y(s)|ds, \tag{2.7}
$$

where $\int_{t_0}^{\infty} a(s)ds < \infty$, $\int_{t_0}^{\infty} b(s)ds < \infty$, $\int_{t_0}^{\infty} k(s)ds < \infty$, $\int_{t_0}^{\infty} q(s)ds < \infty$. *Then, any solution* $y(t) = y(t, t_0, y_0)$ *of (1.2) is bounded on* $[t_0, \infty)$ *and it satisfies*

$$|y(t)| \le h(t)W^{-1}\Big[W(c)+c_2\int_{t_0}^{t}(a(s)+b(s)+b(s)\int_{t_0}^{s}(k(\tau)+q(\tau))d\tau]ds\Big],$$

where $t_0 \le t < b_1$, W, W^{-1} *are the same functions as in Lemma 1.8, and*

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + c_2\int_{t_0}^{t}(a(s) + b(s)$$
$$+b(s)\int_{t_0}^{s}(k(\tau) + q(\tau))d\tau]ds \in \mathrm{domW}^{-1}\Big\}.$$

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (1.1) and (1.2), respectively. By the same argument as in the proof in Theorem 2.2, the solution $z = 0$ of (1.4) is hS. Using the nonlinear variation of constants formula, the hS condition of $x = 0$ of (1.1), together with (2.6) and (2.7), we have

$$|y(t)| \le c_1|y_0|h(t)\,h(t_0)^{-1} + \int_{t_0}^{t} c_2h(t)h(s)^{-1}\Big(a(s)|y(s)| + b(s)w(|y(s)|)$$
$$+b(s)\int_{t_0}^{s} k(\tau)|y(\tau)|d\tau + b(s)\int_{t_0}^{s} q(\tau)|y(\tau)|d\tau\Big)ds.$$

It follows from (H2) and (H3) that

$$|y(t)| \le c_1|y_0|h(t)\,h(t_0)^{-1} + \int_{t_0}^{t} c_2h(t)\Big(a(s)\frac{|y(s)|}{h(s)} + b(s)w(\frac{|y(s)|}{h(s)})$$
$$+b(s)\int_{t_0}^{s}(k(\tau) + q(\tau))\frac{|y(\tau)|}{h(\tau)}d\tau\Big)ds.$$

Set $u(t) = |y(t)||h(t)|^{-1}$. Then, by Corollary 1.10, we have

$$|y(t)| \le h(t)W^{-1}\Big[W(c) + c_2\int_{t_0}^{t}[a(s) + b(s) + b(s)\int_{t_0}^{s}(k(\tau) + q(\tau))d\tau]ds\Big],$$

where $c = c_1|y_0|h(t)\,h(t_0)^{-1}$. Thus, any solution $y(t) = y(t, t_0, y_0)$ of (1.2) is bounded on $[t_0, \infty)$, and so the proof is complete. □

Remark 2.5. Letting $b(s) = 0$ in Theorem 2.4, we obtain the same result as that of Theorem 3.3 in [11].

Remark 2.6. Letting $w(u) = u$ and $h(t, y(t), Ty(t)) = 0$ in Theorem 2.4, we obtain the same result as that of Theorem 3.1 in [10].

THEOREM 2.7. *Let* $a, b, c, k, q, u, w \in C(\mathbb{R}^+)$. *Suppose that (H1), (H2), (H3), and* g *in (1.2) satisfies*

$$|g(t, y(t))| \le a(t)w(|y(t)|) + b(t)\int_{t_0}^{t} k(s)w(|y(s)|)ds \tag{2.8}$$

and

$$|h(t, y(t), Ty(t))| \le c(t)(|y(t)| + |Ty(t)|), |Ty(t)| \le \int_{t_0}^{t} q(s)|y(s)|ds, \tag{2.9}$$

where $t \ge t_0 \ge 0$, $\int_{t_0}^{\infty} a(s)ds < \infty$, *and* $\int_{t_0}^{\infty} b(s)ds < \infty$, $\int_{t_0}^{\infty} c(s)ds < \infty$, $\int_{t_0}^{\infty} k(s)ds < \infty$, *and* $\int_{t_0}^{\infty} q(s)ds < \infty$. *Then, any solution* $y(t) = y(t, t_0, y_0)$ *of (1.2) is bounded on* $[t_0, \infty)$ *and*

$$\begin{aligned} |y(t)| \le h(t)W^{-1}\Big[&W(c) + c_2\int_{t_0}^{t}(c(s) + \int_{t_0}^{s}(a(\tau) \\ &+ b(\tau)\int_{t_0}^{\tau} k(r)dr)d\tau + c(s)\int_{t_0}^{s} q(\tau)d\tau)ds\Big], \end{aligned}$$

where W, W^{-1} *are the same functions as in Lemma 1.8, and*

$$\begin{aligned} b_1 = \sup\Big\{t \ge t_0 : W(c) + &\quad c_2\textstyle\int_{t_0}^{t}(c(s) + \int_{t_0}^{s}(a(\tau) + b(\tau)\int_{t_0}^{\tau} k(r)dr)d\tau \\ &+ c(s)\textstyle\int_{t_0}^{s} q(\tau)d\tau)ds \in \mathrm{domW^{-1}}\Big\}. \end{aligned}$$

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (1.1) and (1.2), respectively. By the same argument as in the proof in Theorem 2.2, the solution $z = 0$ of (1.4) is hS. By Lemma 1.4, Lemma 1.5, the hS condition of $x = 0$ of (1.1), together with (2.8) and (2.9), we have

$$\begin{aligned} |y(t)| \le c_1|y_0|h(t)\,h(t_0)^{-1} + \int_{t_0}^{t} c_2h(t)h(s)^{-1}\Big(\int_{t_0}^{s}(a(\tau)w(|y(\tau)|) \\ + b(\tau)\int_{t_0}^{\tau} k(r)w(|y(r)|)dr)d\tau + c(s)(|y(s)| + \int_{t_0}^{s} q(\tau)|y(\tau)|d\tau)\Big)ds. \end{aligned}$$

Using the assumptions (H2) and (H3), we obtain

$$\begin{aligned} |y(t)| \le c_1|y_0|h(t)\,h(t_0)^{-1} + \int_{t_0}^{t} c_2h(t)\Big(c(s)\frac{|y(s)|}{h(s)} \\ + \int_{t_0}^{s}(a(\tau)w(\frac{|y(\tau)|}{h(\tau)}) + b(\tau)\int_{t_0}^{\tau} k(r)w(\frac{|y(r)|}{h(r)})dr)d\tau \end{aligned}$$

$$+ c(s)\int_{t_0}^{s} q(\tau)\frac{|y(\tau)|}{h(\tau)}d\tau\Big)ds.$$

Set $u(t) = |y(t)||h(t)|^{-1}$. Then, it follows from Lemma 1.11 that we have

$$|y(t)| \le h(t)W^{-1}\Big[W(c) + c_2\int_{t_0}^{t}(c(s) + \int_{t_0}^{s}(a(\tau)$$
$$+ b(\tau)\int_{t_0}^{\tau} k(r)dr)d\tau + c(s)\int_{t_0}^{s} q(\tau)d\tau)ds\Big],$$

where $c = c_1|y_0|\,h(t_0)^{-1}$. From the above estimation, we obtain the desired result. Thus, the theorem is proved. □

Remark 2.8. Letting $c(t) = 0$ in Theorem 2.7, we obtain the similar result as that of Theorem 3.6 in [9].

Theorem 2.9. *Let* $a, b, c, k, q, u, w \in C(\mathbb{R}^+)$. *Suppose that (H1), (H2), (H3), and* g *in (1.2) satisfies*

$$\int_{t_0}^{s} |g(\tau, y(\tau))|d\tau \le a(s)w(|y(s)|) + b(s)\int_{t_0}^{s} k(\tau)w(|y(\tau)|)d\tau, \tag{2.10}$$

and

$$|h(t, y(t), Ty(t))| \le c(t)(|y(t)| + |Ty(t)|), |Ty(t)| \le \int_{t_0}^{t} q(s)|y(s)|ds, \tag{2.11}$$

where $\int_{t_0}^{\infty} a(s)ds < \infty$, $\int_{t_0}^{\infty} b(s)ds < \infty$, $\int_{t_0}^{\infty} c(s)ds < \infty$, $\int_{t_0}^{\infty} k(s)ds < \infty$, *and* $\int_{t_0}^{\infty} q(s)ds < \infty$. *Then, any solution* $y(t) = y(t, t_0, y_0)$ *of (1.2) is bounded on* $[t_0, \infty)$ *and it satisfies*

$$|y(t)| \le h(t)W^{-1}\Big[W(c) + c_2\int_{t_0}^{t}(a(s) + c(s) + b(s)\int_{t_0}^{s} k(\tau)d\tau$$
$$+ c(s)\int_{t_0}^{s} q(\tau)d\tau)ds\Big],$$

where $t_0 \le t < b_1$, W, W^{-1} *are the same functions as in Lemma 1.8, and*

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + c_2\int_{t_0}^{t}(a(s) + c(s)$$
$$+b(s)\int_{t_0}^{s} k(\tau)d\tau + c(s)\int_{t_0}^{s} q(\tau)d\tau)ds \in \mathrm{domW^{-1}}\Big\}.$$

Proof. Let $x(t) = x(t,t_0,y_0)$ and $y(t) = y(t,t_0,y_0)$ be solutions of (1.1) and (1.2), respectively. By the same argument as in the proof in Theorem 2.2, the solution $z = 0$ of (1.4) is hS. Using the nonlinear variation of constants formula, the hS condition of $x = 0$ of (1.1), together with (2.10) and (2.11), we have

$$|y(t)| \leq c_1|y_0|h(t)\,h(t_0)^{-1} + \int_{t_0}^{t} c_2 h(t)h(s)^{-1}\Big((a(s)w(|y(s)|)$$
$$+ c(s)|y(s)| + b(s)\int_{t_0}^{s} k(\tau)w(|y(\tau)|)d\tau + c(s)\int_{t_0}^{s} q(\tau)|y(\tau)|d\tau\Big)ds.$$

Using (H2) and (H3), we obtain

$$|y(t)| \leq c_1|y_0|h(t)\,h(t_0)^{-1} + \int_{t_0}^{t} c_2 h(t)\Big(a(s)w(\frac{|y(s)|}{h(s)}) + c(s)\frac{|y(s)|}{h(s)}$$
$$+ b(s)\int_{t_0}^{s} k(\tau)w(\frac{|y(\tau)|}{h(\tau)})d\tau + c(s)\int_{t_0}^{s} q(\tau)\frac{|y(\tau)|}{h(\tau)}d\tau\Big)ds.$$

Set $u(t) = |y(t)||h(t)|^{-1}$.Then, an application of Lemma 1.9 yields

$$|y(t)| \leq h(t)W^{-1}\Big[W(c) + c_2\int_{t_0}^{t}(a(s) + c(s) + b(s)\int_{t_0}^{s} k(\tau)d\tau$$
$$+ c(s)\int_{t_0}^{s} q(\tau)d\tau)ds\Big],$$

where $c = c_1|y_0|h(t)\,h(t_0)^{-1}$. Then, any solution $y(t) = y(t,t_0,y_0)$ of (1.2) is bounded on $[t_0,\infty)$, and so the proof is complete. ☐

Remark 2.10. Letting $c(t) = 0$ in Theorem 2.9, we obtain the same result as that of Theorem 3.2 in [8].

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